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Solomon's Second Conjecture: A Proof for Local Hereditary Orders in Central Simple Algebras

MARLEEN DENERT

Bredenakkerstraat 5, 9070 Destelbergen, Belgium

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1. INTRODUCTION

Solomon's second conjecture concerns partial zeta functions of local orders. The exact definitions of orders and ideals can be found in [R], and we use the partial zera functions, as defined in [BR]. We briefly recall the definitions.

Let R_v be a discrete valuation ring in a local field K with valuation v. A local order Θ is an R_v-order in a central simple K-algebra $A = M_n(D)$. If $n = 1$ the valuation v has a unique extension v_D in the skew field D. We denote with Δ the unique maximal R_v-order in D and with $\not{p}$ the maximal ideal in Δ. Then $\#(\Delta/\not{p}) = q$ is finite. We fix a uniformizing element π of Δ, i.e., $v_D(\pi) = 1$. If $n \neq 1$ the local orders are not so easy to describe. The maximal R_v-orders Λ in A are isomorphic with $M_n(\Delta)$.

The partial zeta functions of a local order Θ correspond to a choice of two left Θ-ideals $\mathscr{L}$, $\mathscr{M}$ in A (cf. [BR, p. 137]):

$$Z_{\mathscr{L},\mathscr{M}}(s) = \sum_{X \cong \mathscr{L} \text{ and } X \subseteq \mathscr{M}} (\mathscr{M} : X)^{-s}, \qquad \Re(s) > 1.$$

This zeta function $Z_{\mathscr{L},\mathscr{M}}(s)$ depends only on the isomorphism class of $\mathscr{L}$ and $\mathscr{M}$. Since the local class number of maximal orders is 1, there is only one partial zeta function which coincides with the (total) zeta function, denoted $\zeta_\Lambda(s)$.

Solomon's first conjecture states that a partial zeta function only differs a polynomial from $\zeta_\Lambda(s)$:

$$Z_{\mathscr{L},\mathscr{M}}(s)/\zeta_\Lambda(s) \in \mathbb{Z}[q, q^{-s}].$$

This was proved in 1981 by C. J. Bushnell and I. Reiner (cf. [BR, p. 147]).

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For his second conjecture, Solomon considered a set $\{\mathscr{L}_i \mid 1 \leqslant i \leqslant h\}$ representing the isomorphism classes of left Θ-ideals in A (h is the local class number of Θ). He arranged the partial zeta functions in an $h \times h$ matrix $A = (Z_{\mathscr{L}_i, \mathscr{L}_j}(s))_{1 \leqslant i, j \leqslant h}$. For this matrix he showed that the inverse matrix $A^{-1} \in M_h(\mathbb{Z}[q, q^{-s}])$ (cf. [S]).

From this fact and the first conjecture he suggested that

$$\det(A)^{-1} = \pm \prod_j (1 - q^{a_j - ns})^{b_j} \qquad \text{with} \quad a_j, b_j \in \mathbb{N} \quad \text{and} \quad b_j \neq 0.$$

In this paper we restrict our discussion to an important class of local orders, namely the hereditary orders. The structure theorem of Harada and Brumer provides that a hereditary R_v-order Θ is determined, up to isomorphism, by its local type r and local invariants $\eta = (n_1, ..., n_r)$ with $n_j \neq 0$ and $\sum_{1 \leqslant j \leqslant r} n_j = n$ (cf. [R, Theorem 39.14]. We denote this hereditary order Θ^η. The integers n_j indicate a separation of the matrix ring $M_n(\Delta)$ into blocks of size n_j (cf. Section 2). In fact the maximal orders can be viewed as hereditary orders with local type $r = 1$. We first improve Solomon's first conjecture for these orders. This result is based on a better description of the partial zeta function, using normal forms. As a direct corollary we can prove the second conjecture for the hereditary orders. Moreover we obtain all the candidate values for $a_j : a_j = j$, $0 \leqslant j < n$. In the second section we further calculate b_j. This involves a lot of technical calculations, using combinatorics. The combinatorial lemmas are proved in an appendix. We obtain

$$b_j = \binom{j + r - 1}{r - 1}.$$

2. The Proof of Solomon's Second Conjecture for Hereditary Orders Θ^η.

The structure of hereditary orders is well known by the Harada–Brumer theorem. To obtain a more useful description of the orders and their ideals, we introduce the following notation:

* We use Greek letters η, κ, λ to denote r-tuples of natural numbers, f.i., the local invariants of $\Theta^\eta : \eta = (n_1, ..., n_r)$.

* The kth partial sum ($k \leqslant r$) of an r-tuple η is abbreviated $S_\eta(k) = \sum_{1 \leqslant i \leqslant k} n_i$. The local invariants η satisfy $S_\eta(r) = n$. We denote $\mathscr{T}_r(n) = \{r\text{-tuples } \kappa \mid S_\kappa(r) = n\}$.

$*$ We frequently use a map η^*, associated to $\eta \in \mathcal{T}_r(n)$ and defined

$$\eta^*: \{1, ..., n\} \to \{1, ..., r\} \qquad \text{such that} \quad S_\eta(\eta^*(i)-1) < i \leqslant S_\eta(\eta^*(i)).$$

η^* is called the block map of η, since i belongs to the block $\eta^*(i)$.

The structure theorem for hereditary orders (cf. [R, Theorem 39.14]) can be reformulated as

$$\Theta^\eta = \{x \in M_n(\Delta) \mid v_D(x_{i,j}) > 0 \text{ if } \eta^*(i) > \eta^*(j)\}.$$

We now determine a set of representatives for the isomorphism classes of left Θ^η-ideals in A. The isomorphism classes of indecomposable left Θ-lattices are represented by column matrices I_m, $m = 1, ..., r$. With [R, Theorem 39.23] reformulated these are described as

$$I_m = \{x \in M_{n\times 1}(\Delta) \mid v_D(x_i) > 0 \text{ if } \eta^*(i) > m\}.$$

A left Θ-ideal $\mathcal{L}$ in A is a full Θ^η-lattice (i.e., $K\mathcal{L} = A$). $\mathcal{L}$ can be constructed by arranging the elements of n column matrices I_m into $n \times n$ matrices:

$$\mathcal{L} = (I_{m_1}, ..., I_{m_n}) = \{x \in M_n(\Delta) \mid v_D(x_{i,j}) > 0 \text{ if } \eta^*(i) > m_j\}.$$

We are interested in isomorphism classes of such ideals, i.e., representatives under right multiplication with elements of A^* (the units of A), which can permute the columns in $M_n(\Delta)$. A set of representatives for the isomorphism classes of left Θ-ideals in A is thus given by

$$\mathcal{L} = (I_{m_1}, ..., I_{m_n}) \qquad \text{with} \quad m_1 \leqslant m_2 \leqslant \cdots \leqslant m_n.$$

Define the r-tuple $\kappa = (k_1, ..., k_r) \in \mathcal{T}_r(n)$, where $k_t = \#\{j \mid m_j = t\}$; then $\mathcal{L} = \mathcal{L}_\kappa$ can be described as

$$\mathcal{L}_\kappa = \{x \in M_n(\Delta) \mid v_D(x_{i,j}) > 0 \text{ if } \eta^*(i) > \kappa^*(j)\}.$$

LEMMA 1. *The total class number* $h = \#\mathcal{T}_r(n) = \binom{n+r-1}{r-1}$.

Proof. We only remark that every element $\kappa = (k_1, ..., k_r) \in \mathcal{T}_r(n)$ can be viewed as a set with n elements, $\{1, ..., 1, 2, ..., 2, ..., r, ..., r\}$, containing k_i elements i. Counting the number of such sets yields the theorem. ∎

For each pair of left Θ-ideals $\mathcal{L}_\kappa$, $\mathcal{L}_\lambda$ in A we defined a partial zeta function, which we briefly denote

$$Z_{\mathcal{L}_\lambda, \mathcal{L}_\kappa}(s) = Z_{\lambda,\kappa}(s).$$

As in [BR, p. 138] we introduce the notation $\{\mathscr{L} : \mathscr{M}\} = \{x \in M_n(D) \mid \mathscr{L}x \subseteq \mathscr{M}\}$. We write X^* for the unit group of X and $\Theta_{\mathscr{L}} = \{\mathscr{L} : \mathscr{L}\}$ for the right order of $\mathscr{L}$. With this notation,

$$Z_{\mathscr{L},\mathscr{M}}(s) = (\mathscr{M} : \mathscr{L})^{-s} \sum_{x \in \Theta_{\mathscr{L}}^* \setminus \{\mathscr{L} : \mathscr{M}\} \cap A^*} \|x\|^s, \qquad \Re(s) > 1. \tag{1}$$

We have to determine $\{\mathscr{L}_\lambda : \mathscr{L}_\kappa\}$; this is done in the following lemma:

LEMMA 2. $\{\mathscr{L}_\lambda : \mathscr{L}_\kappa\} = \{a \in M_n(\Delta) \mid v_D(a_{i,j}) > 0$ *if* $\lambda^*(i) > \kappa^*(j)\}$.

Proof. We consider the Δ-basis $\{x_{k,l} \mid 1 \leqslant k, l \leqslant n\}$ of $\mathscr{L}_\lambda$ defined by

$$x_{k,l} = (\delta_{ik}\delta_{jl}\pi^{v(k,l)}) \qquad \text{with} \quad v(k,l) = \begin{cases} 1 & \text{if} \quad \eta^*(k) > \lambda^*(l) \\ 0 & \text{if} \quad \eta^*(k) \leqslant \lambda^*(l). \end{cases}$$

Then $a \in \{\mathscr{L}_\lambda : \mathscr{L}_\kappa\}$ if and only if $x_{k,l}a \in \mathscr{L}_\kappa$ for every k, l.

We calculate

$$(x_{k,l}\, a)_{i,j} = \delta_{ki} a_{l,j} \pi^{v(k,l)} = \begin{cases} 0 & \text{if} \quad i \neq k \\ a_{l,j}\, \pi^{v(k,l)} & \text{if} \quad i = k, \end{cases}$$

and thus

$$\begin{aligned} x_{k,l}\, a \in \mathscr{L}_\kappa &\Leftrightarrow v_D(a_{l,j}) + v(k,l) > 0 \qquad \text{for} \quad \eta^*(k) > \kappa^*(j) \\ &\Leftrightarrow a \in M_n(\Delta) \qquad \text{and} \qquad v_D(a_{l,j}) > 0 \\ &\qquad \text{if} \quad \kappa^*(j) < \eta^*(k) \leqslant \lambda^*(l) \quad \text{for some } k \\ &\Leftrightarrow a \in M_n(\Delta) \qquad \text{and} \qquad v_D(a_{l,j}) > 0 \qquad \text{if} \quad \kappa^*(j) < \lambda^*(l). \end{aligned}$$

This settles the proof. ∎

As a direct corollary we obtain that the right order of $\mathscr{L}_\lambda$ is the hereditary order Θ^λ.

The further determination of $Z_{\lambda,\kappa}(s)$ is inspired by the calculation of the zeta function $\zeta_\Lambda(s)$ of a maximal order $\Lambda \cong M_n(\Delta)$. We recall the following from [BR, Chap. 3.3]:

LEMMA 3.

$$\zeta_\Lambda(s) = \prod_{0 \leqslant j \leqslant n-1} (1 - q^{j-ns})^{-1}. \tag{2}$$

Proof. applying formula (1) to the maximal order $\Lambda = \Theta^{(n)}$ we find

$$\zeta_\Lambda(s) = \sum_{x \in M_n(\Delta)^* \setminus M_n(\Delta) \cap A^*} \|x\|^s.$$

The fact that every element $x \in M_n(\Delta) \cap A^*$ has a unique "Hermite-normal form," obtained by left multiplication with $M_n(\Delta)^*$, yields

$$\zeta_A(s) = \sum_{x \text{ a Hermite-normal form}} \|x\|^s.$$

The Hermite-normal form x is a reduced upper-triangular matrix. The diagonal entries are of the form $x_{j,j} = \pi^{m_j}$. We say that x belongs to $(m_j) \in \mathbb{N}^n$. Moreover we find that $\|x\| = q^{-(m_1 + \cdots + m_n)n}$ and we recall from [BR, D] that the number of Hermite-normal forms in $M_n(\Delta) \cap A^*$ belonging to (m_j) is

$$q^{0m_1 + 1m_2 + \cdots + (n-1)m_n}.$$

This yields

$$\begin{aligned}\zeta_A(s) &= \sum_{(m_j) \in \mathbb{N}^n} q^{0m_1 + 1m_2 + \cdots + (n-1)m_n} \prod_{1 \leqslant j \leqslant n} (q^{-ns})^{m_j} \\ &= \sum_{(m_j) \in \mathbb{N}^n} \prod_{1 \leqslant j \leqslant n} (q^{j-1-ns})^{m_j} = \prod_{1 \leqslant j \leqslant n} (1 - q^{j-1-ns})^{-1}. \quad \blacksquare\end{aligned}$$

The calculation of $\zeta_A(s)$ depends on a good defined Hermite-normal form. For a hereditary order Θ^λ we showed in [D p. 36], that every element $x \in \Theta^\lambda \cap A^*$ can be reduced to some "normal form" by left multiplication with $(\Theta^\lambda)^*$. This reduction is called the "Θ^λ-normal form" of x. It is easy to extend this reduction to all elements $x \in M_n(\Delta) \cap A^*$.

The main difference between the Hermite-normal form and the Θ^λ-normal form of x is that the latter is no longer an upper-triangular matrix (ie., $x_{i,j} = 0$ if $j < i$), but rather depends on a permutation $\sigma \in S_n$ such that $x_{i,j} = 0$ if $j < \sigma(i)$.

We extend the definition of Θ^λ-normal form to $x \in M_n(\Delta) \cap A^*$:

* Denote by $S^{(\lambda)} = \{\sigma \in S_n \mid \lambda^*(i) = \lambda^*(i') \text{ and } i < i' \Rightarrow \sigma(i) < \sigma(i')\}$ the set of (λ)-admissible permutations.

* $x \in M_n(\Delta) \cap A^*$ is an Θ^λ-normal form, belonging to $\sigma \in S^{(\lambda)}$ and $(m_j) \in \mathbb{N}^n$ if and only if

$$x_{i,\sigma(i)} = \pi^{m_{\sigma(i)}}$$

$$x_{i,j} = 0 \qquad \text{if } \ j < \sigma(i)$$

$$x_{i,j} \in \begin{cases} \Delta/\mathfrak{p}^{m_j} & \text{if (s1)} \\ \Delta/\mathfrak{p}^{m_j+1} & \text{if (s2),} \end{cases}$$

where (s1) and (s2) are the following conditions on (i, j):

$$\text{(s1)} \quad j > \sigma(i) \qquad \text{and} \qquad \sigma^{-1}(j) > i.$$

$$\text{(s2)} \quad j > \sigma(i) \qquad \text{and} \qquad \sigma^{-1}(j) < i.$$

In order to calculate the partial zeta functions we have to restrict to Θ^{λ}-normal forms $x \in \{\mathscr{L}_\lambda : \mathscr{L}_\kappa\} \subset M_n(\Delta)$. In view of Lemma 2 we add a condition on (i, j):

$$\text{(s3)} \quad \kappa^*(j) < \lambda^*(i).$$

This yields the condition $v_D(x_{i,j}) > 0$ if (s3), and if $x \in \{\mathscr{L}_\lambda : \mathscr{L}_\kappa\}$ belongs to (m_j) then

$$\text{(m1)} \quad m_j > 0 \qquad \text{if} \quad \kappa^*(j) < \lambda^*(\sigma^{-1}(j)).$$

As for Hermite-normal forms we find that the module of an Θ^{λ}-normal form x belonging to (m_j) is $\|x\| = q^{-(m_1 + \cdots + m_n)n}$. For the number of Θ^{λ}-normal forms in $\{\mathscr{L}_\lambda : \mathscr{L}_\kappa\} \cap A^*$ belonging to $\sigma \in S^{(\lambda)}$ and $(m_j) \in \mathbb{N}^n$ satisfying (m1), we obtain

$$q^{N_\sigma(\lambda, \kappa) + 0m_1 + 1m_1 + \cdots + (n-1)m_n}.$$

It is important to remark that the correction $N_\sigma(\lambda, \kappa)$ depends on $\sigma \in S^{(\lambda)}$, λ, κ but not on (m_j). Namely $N_\sigma(\lambda, \kappa) = N_\sigma^+(\lambda, \kappa) - N_\sigma^-(\lambda, \kappa)$ with

$$N_\sigma^+(\lambda, \kappa) = \#\{(i, j) \mid \text{(s2) holds but (s3) does not}\}$$

$$N_\sigma^-(\lambda, \kappa) = \#\{(i, j) \mid \text{(s1) and (s3) hold}\}.$$

Before calculating $\mathbb{Z}_{\lambda,\kappa}(s)$ we still need the following lemma:

LEMMA 4. *For every subset* $I \subseteq \{1, ..., n\}$,

$$\sum_{\substack{(m_j) \in \mathbb{N}^n \\ m_j > 0 \text{ if } j \in I}} \prod_{1 \leqslant j \leqslant n} (f_j(s))^{m_j} = \prod_{1 \leqslant j \leqslant n} (1 - f_j(s))^{-1} \prod_{j \in I} f_j(s).$$

Proof. Remark first that $\sum_{m_j \in \mathbb{N}} (f_j(s))^{m_j} = (1 - f_j(s))^{-1}$ (in the domain of convergence). For any function $g(s)$ we split up the sum as

$$\sum_{\substack{(m_j) \\ m_j > 0 \text{ if } j \in I}} g(s) = \sum_{(m_j)} g(s) - \sum_{k \in I} \sum_{\substack{(m_j) \\ m_k = 0}} g(s) + \cdots + (-1)^{\# I} \sum_{\substack{(m_j) \\ m_k = 0 \text{ if } k \in I}} g(s)$$

and thus for $g(s)=\prod_{1\leqslant j\leqslant n}(f_j(s))^{m_j}$ we obtain

$$\sum_{\substack{(m_j)\\ m_j>0 \text{ if } j\in I}} g(s)=\prod_{1\leqslant j\leqslant n}(1-f_j(s))^{-1}$$

$$\cdot\left(1-\sum_{k\in I}(1-f_k(s))+\cdots+(-1)^{\# I}\prod_{k\in I}(1-f_k(s))\right)$$

$$=\prod_{1\leqslant j\leqslant n}(1-f_j(s))^{-1}\prod_{j\in I}f_j(s).$$

This settles the proof. ▮

We can now formulate a refinement of Solomon's first conjecture for the partial zeta functions of hereditary orders:

THEOREM 5. $Z_{\lambda,\kappa}(s)\ (\mathscr{L}_\kappa:\mathscr{L}_\lambda)^s/\zeta_\Lambda(s)=f_{\lambda,\kappa}(s)\in\mathbb{Z}[q,q^{-ns}]$. *Moreover,*

$$f_{\lambda,\kappa}(s)=\sum_{\sigma\in S^{(\lambda)}}q^{N_\sigma(\lambda,\kappa)}\prod_{j\in I_\sigma(\lambda,\kappa)}q^{j-1-ns}, \tag{3}$$

where $I_\sigma(\lambda,\kappa)=\{j\mid\kappa^*(j)<\lambda^*(\sigma^{-1}(j))\}$.

Proof. From (1) we obtain that $Z_{\lambda,\kappa}(s)=(\mathscr{L}_\kappa:\mathscr{L}_\lambda)^{-s}g(s)$ with

$$g(s)=\sum_{x\in(\Theta^\lambda)^*\backslash\{\mathscr{L}_\lambda:\mathscr{L}_\kappa\}\cap\Lambda^*}\|x\|^s=\sum_{\Theta^\lambda\text{-normal form } x \text{ in } \{\mathscr{L}_\lambda:\mathscr{L}_\kappa\}}\|x\|^s$$

$$=\sum_{\sigma\in S^{(\lambda)}}\sum_{\substack{(m_j)\in\mathbb{N}^n\\ m_j>0 \text{ if } j\in I_\sigma(\lambda,\kappa)}}q^{N_\sigma(\lambda,\kappa)}\prod_{1\leqslant j\leqslant n}(q^{j-1-ns})^{m_j}$$

$$=\sum_{\sigma\in S^{(\lambda)}}q^{N_\sigma(\lambda,\kappa)}\sum_{\substack{(m_j)\\ m_j>0 \text{ if } j\in I_\sigma(\lambda,\kappa)}}\prod_{1\leqslant j\leqslant n}(q^{j-1-ns})^{m_j}.$$

By applying Lemma 4 with $I=I_\sigma(\lambda,\kappa)$ and $f_j(s)=q^{j-1-ns}$ we obtain

$$g(s)=\sum_{\sigma\in S^{(\lambda)}}q^{N_\sigma(\lambda,\kappa)}\prod_{0\leqslant j\leqslant n-1}(1-q^{j-ns})^{-1}\prod_{j\in I_\sigma(\lambda,\kappa)}q^{j-1-ns}$$

$$=\zeta_\Lambda(s)\cdot f_{\lambda,\kappa}(s)$$

with $f_{\lambda,\kappa}(s)$ given above. ▮

The proof of Solomon's second conjecture only depends on the refinement, obained in Theorem 5, $f_{\lambda,\kappa}(s)\in\mathbb{Z}[q,q^{-ns}]$.

Since $\{\mathscr{L}_\lambda\mid\lambda\in\mathscr{T}_r(n)\}$ is a set of representatives for the isomorphism classes of left Θ^n-ideals, we must consider the matrix $A = (Z_{\lambda,\kappa}(s))_{\lambda,\kappa\in\mathscr{T}_r(n)}$ and calculate the determinant.

COROLLARY 6. $\det(A)^{-1} = \pm\prod_{0\leqslant j\leqslant n-1}(1-q^{j-ns})^{b_j}$ *with* $0\leqslant b_j\leqslant h$.

Proof. We denote $X=q^{-s}$ and $Y=X^n$. Solomon showed that $\det(A)^{-1}\in\mathbb{Z}[q, X]$. From Theorem 5 we obtain

$$\det(A)=\det((\mathscr{L}_\kappa:\mathscr{L}_\lambda)^{-s}\zeta_A(s)\,f_{\lambda,\kappa}(s))=(\zeta_A(s))^h\det(f_{\lambda,\kappa}(s)) \qquad (*)$$

(namely for every permutation P of $\mathscr{T}_r(n)\prod_{\lambda\in\mathscr{T}_r(n)}(\mathscr{L}_\lambda:\mathscr{L}_{P(\lambda)})=1$). Expression (2) for $\zeta_A(s)$ and expression (3) for $f_{\lambda,\kappa}(s)\in\mathbb{Z}[q, Y]$ remain. It follows that

$$\det(A)^{-1}\in\mathbb{Z}[q][X]\cap\mathbb{Z}[q](Y)=\mathbb{Z}[q, Y].$$

We rewrite $(*)$ and consider the factors as elements of $\mathbb{Z}[q, Y]$:

$$\det(A)^{-1}\det(f_{\lambda,\kappa}(s))=\prod_{0\leqslant j\leqslant n-1}(1-q^jY)^h. \qquad (**)$$

Remark now that the factors $(1-q^jY)$ are irreducible in $\mathbb{Z}[q, Y]$, namely

$$(1-q^jY)=f(q, Y)\,g(q, Y) \quad \text{yields} \quad \deg_Y f=0 \quad \text{and} \quad \deg_Y g=1,$$

and thus

$$(1-q^jY)=f_0(q)(g_0(q)+g_1(q)\,Y).$$

We conclude that

$$f_0(q)\,g_0(q)=1 \qquad \text{so} \quad f(q, Y)=f_0(q) \text{ is a unit in } \mathbb{Z}[q, Y].$$

Since $\mathbb{Z}[q, Y]$ is a unique factorization ring we conclude from $(**)$ that $\det(A)^{-1}$ has the required form. ∎

Moreover as candidate values for a_j we obtain $a_j=j$, $0\leqslant j<n$. In the next section we will calculate b_j explicitly.

3. DETERMINATION OF b_j

As in the proof of Corollary 6 we introduce the notation $Y=q^{-ns}$. It is also useful to introduce the matrix of polynomials

$$F=(f_{\lambda,\kappa}(s))_{\lambda,\kappa\in\mathscr{T}_r(n)}\in M_h(\mathbb{Z}[q, Y]).$$

It is quite obvious from the proof of Corollary 6 that we can calculate b_j from $\det(F)$, which is an expression in $\mathbb{Z}[q, Y]$:

$$\det(F) = \det(f_{\lambda,\kappa}(s)) = \prod_{0 \leqslant j \leqslant n-1} (1 - q^j Y)^{h - b_j}.$$

We now consider $\det(F)$ as a polynomial in Y with coefficients in $\mathbb{Z}[q]$. The coefficient of Y determines b_j completely since

$$\det(F) = 1 - Y\left\{\sum_{0 \leqslant j \leqslant n-1} (h - b_j) q^j\right\} + \cdots. \tag{4}$$

Therefore we can restrict our discussion to the calculation of $\det(\bar{F}) = \det(F) \bmod Y^2$. The entries of F are given in Theorem 5:

$$f_{\lambda,\kappa}(s) = \sum_{\sigma \in S^{(\lambda)}} q^{N_\sigma(\lambda,\kappa)} \prod_{j \in I_\sigma(\lambda,\kappa)} (q^{j-1} Y).$$

To determine $\overline{f_{\lambda,\kappa}(s)} = f_{\lambda,\kappa}(s) \bmod Y^2$ we restrict ourselves to $\sigma \in S^{(\lambda)}$ with $\# I_\sigma(\lambda, \kappa) \leqslant 1$. To simplify the notation we set $I_\sigma(\lambda, \kappa) = \{0\}$ if this set is empty. From now on we fix the following notation: $I_\sigma(\lambda, \kappa) = \{b\}$ with $0 \leqslant b \leqslant n$ and $b = \sigma(i_o)$. Moreover we denote $\lambda^*(i_o) = M_\sigma$ and $\kappa^*(b) = m_\sigma < r$. If $b = 0$ then we define $i_o = n+1$, $M_\sigma = r+1$, and $m_\sigma = r$. We write $\overline{f_{\lambda,\kappa}(s)}$ by assembling all the terms corresponding to b.

Denote with

$$C_b(\lambda, \kappa) = \sum_{\substack{\sigma \in S^{(\lambda)} \\ I_\sigma(\lambda,\kappa) = \{b\}}} q^{N_\sigma(\lambda,\kappa)} \in \mathbb{Z}[q], \qquad 0 \leqslant b \leqslant n.$$

Then

$$\overline{f_{\lambda,\kappa}(s)} = C_0(\lambda, \kappa) + Y \sum_{1 \leqslant b \leqslant S_\kappa(r-1)} C_b(\lambda, \kappa) q^{b-1}.$$

In the appendix we proved some combinatorial lemmas to simplify the calculation of $C_b(\lambda, \kappa)$.

LEMMA 7. *Take $\lambda, \kappa \in \mathcal{T}_r(n)$.*

(i) *$S_\kappa(k) = S_\lambda(k) + N$ with $N \geqslant 2$ for some k then $\overline{f_{\lambda,\kappa}(s)} = 0$.*

(ii) *If $S_\kappa(k) = S_\lambda(k) + 1$ for some k then $C_0(\lambda, \kappa) = 0$.*

Proof. This is a direct consequence of Lemma A.1. ∎

In view of this, we introduce an ordering on $\mathcal{T}_r(n)$ such that above the diagonal of $\bar{F}$, a lot of zeros appear:

DEFINITION. $\lambda < \kappa \Leftrightarrow$ there exists $m < r$ such that $S_\lambda(j) = S_\kappa(j)$ for every $j < m$ and $S_\lambda(m) < S_\kappa(m)$.

Not only do we obtain a lot of zeros above the diagonal but we also find that if $\lambda < \kappa$ (above the diagonal) then $\overline{f_{\lambda,\kappa}(s)} \in Y\mathbb{Z}[q]$. So, in order to calculate $\det(\bar{F})$ we only need to calculate the constant term C_0 for the entries under the diagonal. The aim is to define column transformations that transform $\bar{F}$ into an under-triangular matrix. The influences on the diagonal entries are calculated, and finally we obtain $\det(\bar{F})$ as the product of these diagonal entries.

We first investigate some relations between the entries. The following notation is used:

$$[k] = \frac{q^k - 1}{q - 1}, \qquad k \geqslant 0.$$

To $m < r$ and κ with $k_m > 0$ we associate $\kappa_m = (k_1, ..., k_m - 1, ..., k_r + 1) \in \mathscr{T}_r(n)$. The elements above the diagonal $\lambda < \kappa$ for which $S_\kappa(j) - 1 \leqslant S_\lambda(j) \leqslant S_\kappa(j)$ for every $j = 1, ..., r$ are calculated in terms of $C_0(\lambda, \kappa_m)$ in the following theorem:

THEOREM 8. *For* $\lambda < \kappa$ *with* $S_\kappa(j) - 1 \leqslant S_\lambda(j) \leqslant S_\kappa(j)$ *for every* $j = 1, ..., r$, $\overline{f_{\lambda,\kappa}(s)} = Y \sum_{1 \leqslant m < r} [k_m] q^{S_\kappa(m-1)} C_0(\lambda, \kappa_m)$.

Proof. Remark that the contribution of m is 0 if $k_m = 0$. So let us assume now that $k_m > 0$. The essential combinatorial property is proved in Lemma A.6:

$$C_b(\lambda, \kappa) = C_0(\lambda, \kappa_m) \qquad \text{if} \quad \kappa^*(b) = m < r.$$

We obtain

$$\begin{aligned}
\overline{f_{\lambda,\kappa}(s)} &= Y \sum_{1 \leqslant b \leqslant S_\kappa(r-1)} C_b(\lambda, \kappa) q^{b-1} \\
&= Y \sum_{1 \leqslant m < r} C_0(\lambda, \kappa_m) \sum_{b \text{ with } \kappa^*(b) = m} q^{b-1} \\
&= Y \sum_{1 \leqslant m < r} C_0(\lambda, \kappa_m) q^{S_\kappa(m-1)} \sum_{0 \leqslant j \leqslant k_m - 1} q^j \\
&= Y \sum_{1 \leqslant m < r} [k_m] q^{S_\kappa(m-1)} C_0(\lambda, \kappa_m). \quad \blacksquare
\end{aligned}$$

To simplify the description of $\sigma \in S^{(\lambda)}$ we define

DEFINITION. σ has a decent i_1 on B if $i_1 \in B$ and there exists $j \in B$ with $i_1 < j$ and $\sigma(i_1) > \sigma(j)$.

In the next theorem we calculate the diagonal entries of $\bar{F}$:

THEOREM 9. *The diagonal entry of $\bar{F}$ corresponding to $\lambda = \kappa$ is*

$$\overline{f_{\lambda,\kappa}(s)} = 1 + Y \sum_{1 \leqslant m < r} C^r_m(\kappa) \sum_{b \text{ with } \kappa^*(b) = m} q^{b-1}$$

with

$$C^r_m(\kappa) = q \sum_{m+1 \leqslant M \leqslant r} [k_M] \prod_{m < s < M} [k_s + 1].$$

Proof. Let $\sigma \in S^{(\lambda)}$ with $I_\sigma(\lambda, \kappa) = \varnothing$; then $m_\sigma = r$. Lemma A.4 yields $i = \sigma(i)$ for every $i \leqslant S_\lambda(r-1)$. This implies however that σ is the identity for which $N_\sigma(\lambda, \kappa) = 0$, and $C_0(\lambda, \kappa) = 1$ follows.

The calculation of $C_b(\lambda, \kappa)$ with $b > 0$ depends on Lemmas A.7 and A.8. Let $S^{(\lambda)}(b, M)$ denote the set of all $\sigma \in S^{(\lambda)}$ with $I_\sigma(\lambda, \kappa) = \{b\}$ and $M_\sigma = M$. Let $m = \kappa^*(b) < r$. From Lemma A.4 (ii and iii), $\lambda^*(i) = \kappa^*(\sigma(i))$ for $i < S_\lambda(m)$ or $i > i_o$. This determines $\sigma(i)$ completely since $\sigma \in S^{(\lambda)}$ and $\lambda = \kappa$. For $m < s \leqslant M$ fixed we consider now $B_s = \{i \neq i_o \mid \lambda^*(i) \leqslant s \leqslant \kappa^*(\sigma(i))\}$. From Lemma A.8, $\sigma \in S^{(\lambda)}(b, M)$ has at most one descent i_1 on B_s and

$$\# B_s = \begin{cases} k_s + 1 & \text{if } s \neq M \\ k_s & \text{if } s = M. \end{cases}$$

The conditions of Lemma A.7 are satisfied. By starting from σ_0 with no descents on B_s for every $m < s \leqslant M$ and considering the sets S_{σ_o} as in Lemma A.7 we can generate every $\sigma \in S^{(\lambda)}(b, M)$. Moreover σ_o is defined by

$$\sigma_0(i) = \begin{cases} i & \text{if } i < b \text{ or } i_o < i \\ i+1 & \text{if } b \leqslant i < i_o \\ b & \text{if } i = i_o. \end{cases}$$

Since $N^-_{\sigma_o} = 0$ and (i_o, i_o) is the only value of (i, j) contributing to $N^+_{\sigma_o}$ we find $N_{\sigma_o} = 1$. By applying Lemma A.7 we obtain

$$\sum_{\sigma \in S^{(\lambda)}(b, M)} q^{N_\sigma(\lambda, \kappa)} = q \prod_{m < s \leqslant M} [\# B_s] = q \cdot [k_M] \prod_{m < s < M} [k_s + 1]$$

Note from the proof of Lemma A.6(**) that i_o is completely determined by $M = \lambda^*(i_o)$; therefore

$$\{\sigma \in S^{(\lambda)} \mid I_\sigma(\lambda, \kappa) = \{b\}\} = \bigcup_{m < M \leqslant r} S^{(\lambda)}(b, M).$$

The expression for $C_b(\lambda, \kappa)$ follows and only depends on $m = \kappa^*(b)$. ∎

The results of Theorem 8 and the fact that $C_0(\lambda, \lambda) = 1$ (Theorem 9) are sufficient to define column transformations to reduce all the entries above the diagonal ($\lambda < \kappa$) in $\bar{F}$ to zero. This is done in the following theorem.

THEOREM 10. *Consider the following column transformations on* κ, $\kappa \in \mathcal{T}_r(n)$:

(i) *column* $\kappa - \sum_{1 \leqslant m < r} C(\kappa_m) Y \times$ *column* κ_m *with* $C(\kappa_m) = q^{S_\kappa(m-1)}[k_m] = \sum_{b \text{ with } \kappa^*(b) = m} q^{b-1}$.

(ii) *column* $\kappa - \sum_{\kappa' \in \mathcal{T}_\kappa} C(\kappa') Y \times$ *column* κ' *with* $\mathcal{T}_\kappa = \{\kappa' \in \mathcal{T}_r(n) \mid \kappa' < \kappa$ *and* $S_{\kappa'}(j) > S_\kappa(j)$ *for some* $j\}$ *and* $C(\kappa')$, $\kappa' \in \mathcal{T}_r(n)$ *defined inductively as*:

$$\overline{f_{\kappa',\kappa}(Y)} - Y \sum_{1 \leqslant m < r} C(\kappa_m) C_0(\kappa', \kappa_m) - Y \sum_{\substack{\kappa' < \kappa'' \\ \kappa'' \in \mathcal{T}_\kappa}} C(\kappa'') C_0(\kappa', \kappa'') = YC(\kappa').$$

Then the entries above the diagonal in $\bar{F}$ *are reduced to zero.*

Proof. Remark that $C(\kappa_m) = 0$ if $k_m = 0$. These column transformations replace a particular entry $\overline{f_{\lambda,\kappa}(s)}$ by

$$\overline{f_{\lambda,\kappa}(s)} - Y \sum_{1 \leqslant m < r} C(\kappa_m) C_0(\lambda, \kappa_m) - Y \sum_{\kappa' \in \mathcal{T}_\kappa} C(\kappa') C_0(\lambda, \kappa').$$

Since $C_0(\lambda, \kappa)$ does not change by these transformations, the order in which we transform the colums is of no importance.

We check now that all entries above the diagonal, i.e., corresponding to $\lambda < \kappa$, are reduced to zero:

(a) $\lambda < \kappa$ and $S_\kappa(j) - 1 \leqslant S_\lambda(j) \leqslant S_\kappa(j)$ for every j. For $\kappa' \in \mathcal{T}_\kappa$ we deduce $S_{\kappa'}(j) \geqslant S_\lambda(j) + 1$; Lemma 7(ii) yields $C_0(\lambda, \kappa') = 0$. The entry $\overline{f_{\lambda,\kappa}(s)}$ is transformed into

$$\overline{f_{\lambda,\kappa}(s)} - Y \sum_{1 \leqslant m \leqslant r} C(\kappa_m) C_0(\lambda, \kappa_m).$$

From Theorem 8 and the definition of $C(\kappa_m)$ we conclude that this is zero.

(b) $\lambda < \kappa$ and $S_\lambda(j) \leqslant S_\kappa(j)$ for every j but $S_\lambda(j_o) < S_\kappa(j_o) - 1$ for some j_o. From Lemma 7(ii), $\overline{f_{\lambda,\kappa}(s)} = 0$, but the transformations do not change it:

$\to S_\lambda(j_o) + 1 \leqslant S_\kappa(j_o) - 1 \leqslant S_{\kappa_m}(j_o)$, for every m, yields $C_0(\lambda, \kappa_m) = 0$.

$\to \kappa' \in \mathcal{T}_\kappa : S'_\kappa(j) > S_\kappa(j) \geqslant S_\lambda(j)$ yields $C_0(\lambda, \kappa') = 0$.

(c) $\lambda < \kappa$ and $S_\lambda(j) > S_\kappa(j)$ for some j. Since $C_0(\lambda, \kappa') = 0$ for $\kappa' > \lambda$, the column transformations replace the entry by

$$\overline{f_{\lambda,\kappa}(s)} - Y \sum_{1 \leqslant m < r} C(\kappa_m) C_0(\lambda, \kappa_m) - Y \sum_{\kappa' \leqslant \lambda \text{ and } \kappa' \in \mathcal{T}_\kappa} C(\kappa') C_0(\lambda, \kappa').$$

But $\lambda \in \mathcal{T}_\kappa$ and $C_0(\lambda, \lambda) = 1$ (cf. Theorem 9). It follows from the definition of $C(\lambda)$ that this entry is also reduced to zero. ∎

We are now able to calculate $\det(\bar{F})$ if we know how the diagonal entries, corresponding to $\lambda = \kappa$, are influenced by the column transformations, described above. This is done in the following two lemmas.

We first calculate $C_0(\kappa, \kappa_m)$ for $k_m > 0$:

LEMMA 11. *If* $k_m > 0$ *then*

$$C_0(\kappa, \kappa_m) = \prod_{m < s \leqslant r} [k_s + 1].$$

Proof. The proof is similar to the proof of Theorem 9.

Denote $S^{(\kappa)}(m, 0) = \{\sigma \in S^{(\kappa)} \mid I_\sigma(\kappa, \kappa_m) = \varnothing\}$. From Lemma A.4(i), we find $\sigma \in S^{(\kappa)}(m, 0): \sigma(i) = i$ if $i < S_\kappa(m)$. For $s > m$ we consider $B_s = \{i \mid \kappa^*(i) \leqslant s \leqslant \kappa^*_m(\sigma(i))\} = \{i_1 < \cdots < i_k\}$. From lemma A.8, $\# B_s = k_s + 1$ and σ has at most one descent i_1 on B_s. Check that the conditions of Lemma A.7 are satisfied. Let $\sigma_0 \in S^{(\kappa)}(m, 0)$ with no descents on B_s for $m < s \leqslant r$; then σ_0 is the identity, so $N_{\sigma_o}(\kappa, \kappa_m) = 0$. Starting from σ_o we can generate $S^{(\kappa)}(m, 0)$, so we obtain from Lemma A.7

$$C_0(\kappa, \kappa_m) = \sum_{\sigma \in S^{(\kappa)}(m,0)} q^{N_\sigma(\kappa, \kappa_m)} = \prod_{m < s \leqslant r} [k_s + 1]. \quad \blacksquare$$

LEMMA 12. *The column transformations, described in Theorem* 10, *replace a diagonal entry, corresponding to* $\lambda = \kappa$, *by*

$$1 - Y[S_\kappa(r-1)].$$

Proof. Recall from Theorem 9 that a diagonal entry is

$$\overline{f_{\kappa,\kappa}(s)} = 1 + Y \sum_{1 \leqslant m < r} C^r_m(\kappa) \sum_{b \text{ with } \kappa^*(b) = m} q^{b-1}$$

with

$$C^r_m(\kappa) = q \sum_{m+1 \leqslant M \leqslant r} [k_M] \prod_{m < s < M} [k_s + 1].$$

For every $\kappa' \in \mathscr{T}_\kappa$ Lemma 7(ii) yields $C_0(\kappa, \kappa') = 0$. The column transformations described in Theorem 10 transform $C^r_m(\kappa)$ into $D^r_m(\kappa) = C^r_m(\kappa) - C_0(\kappa, \kappa_m)$.

Using lemma 11 we find $D^r_m(\kappa) = C^r_m(\kappa) - \prod_{m < s \leqslant r} [k_s + 1]$.

Claim. $D^r_m(\kappa) = -1$. We prove this by induction on $r > m$:

* $r = m + 1: D^r_m(\kappa) = q[k_r] - [k_r + 1] = -1$.
* Assume that $D^{r-1}_m(\kappa) = -1$.

We split off the term in $C^r_m(\kappa)$ corresponding to r:

$$D^r_m(\kappa) = C^{r-1}_m(\kappa) + q[k_r] \prod_{m<s<r} [k_s+1] - \prod_{m<s\leqslant r} [k_s+1]$$

$$= C^{r-1}_m(\kappa) + \prod_{m<s<r} [k_s+1](q[k_r] - [k_r+1])$$

$$= C^{r-1}_m(\kappa) - \prod_{m<s\leqslant r-1} [k_s+1] = D^{r-1}_m(\kappa) = -1.$$

We conclude that the diagonal entry is replaced by

$$1 - Y \sum_{1\leqslant m<r} \sum_{b \text{ with } \kappa^*(b)=m} q^{b-1} = 1 - Y \sum_{0\leqslant b<S_\kappa(r-1)} q^b.$$

This settles the proof. ∎

Now we can calculate det $(\bar{F})$ and determine the exact values of b_j:

THEOREM 13. *For a local hereditary order in $M_n(D)$ with local type r, Solomon's second conjecture is true and*

$$\det(A)^{-1} = \prod_{0\leqslant j\leqslant n-1} (1-q^jY)^{\binom{j+r-1}{r-1}}.$$

Proof. The determinant of $(\bar{F})$ is the product of the diagonal entries in the reduced matrix:

$$\det(\bar{F}) = \prod_{\kappa\in\mathscr{T}_r(n)} (1 - Y[S_\kappa(r-1)]).$$

From Lemma 1 we find that the number of r-tuples $\kappa\in\mathscr{T}_r(n)$ with $S_\kappa(r-1)=k$ is $\#\mathscr{T}_{r-1}(k) = \binom{k+r-2}{r-2}$. This yields the relation

$$\sum_{0\leqslant k\leqslant n} \binom{k+r-2}{r-2} = \binom{n+r-1}{r-1} = h.$$

We calculate $\det(\bar{F})$ mod Y^2:

$$\det(\bar{F}) = 1 - Y\left\{\sum_{0\leqslant k\leqslant n} \binom{k+r-2}{r-2}[k]\right\}$$

$$= 1 - Y\left\{\sum_{0\leqslant j<n} q^j \sum_{j<k\leqslant n} \binom{k+r-2}{r-2}\right\}$$

$$= 1 - Y\left\{\sum_{0\leqslant j<n} q^j \left[h - \sum_{0\leqslant k\leqslant j} \binom{k+r-2}{r-2}\right]\right\}$$

$$= 1 - Y\left\{\sum_{0\leqslant j<n} q^j \left[h - \binom{j+r-1}{r-1}\right]\right\}.$$

Using (4) the exact values of b_j follow. ∎

4. Appendix

In this section we prove some "combinatorial" lemmas to simplify the calculation of

$$C_b(\lambda, \kappa) = \sum_{\substack{\sigma \in S^{(\lambda)} \\ I_\sigma(\lambda, \kappa) = \{b\}}} q^{N_\sigma(\lambda, \kappa)} \in \mathbb{Z}[q], \qquad 0 \leqslant b \leqslant n.$$

Lemma A.1. *Let $\lambda, \kappa \in \mathscr{T}_r(n)$ such that for some $k : S_\kappa(k) = S_\lambda(k) + N$, $N \geqslant 0$, there exist $i_1, ..., i_N$ with $\lambda^*(i_k) > k > \kappa^*(\sigma(i_k))$. This also yields $\# I_\sigma(\lambda, \kappa) \geqslant N$ for every $\sigma \in S^{(\lambda)}$.*

Proof. Let $\sigma \in S^{(\lambda)}$; then

$$\# \{i \mid \lambda^*(i) \leqslant k\} = S_\lambda(k)$$
$$\# \{i \mid \kappa^*(\sigma(i)) \leqslant k\} = S_\kappa(k) = S_\lambda(k) + N.$$

So there exist $i_1, ..., i_N$ with $\lambda^*(i) > k$ and $\kappa^*(\sigma(i)) \leqslant k$. ∎

From now on we restrict our discussion to $\sigma \in S^{(\lambda)}$ with $I_\sigma(\lambda, \kappa) = \{b = \sigma(i_o)\}$, $0 \leqslant b < n$. Recall that $m_\sigma = \kappa^*(b)$ and $M_\sigma = \lambda^*(i_o)$, with $i_o = n + 1$, $m_\sigma = r$, and $M_\sigma = r + 1$ if $b = 0$.

Lemma A.2. *Let $\lambda, \kappa \in \mathscr{T}_r(n)$; then*

$$M_\sigma > j \geqslant m_\sigma \qquad \textit{for every } j \textit{ satisfying} \quad S_\kappa(j) - 1 = S_\lambda(j).$$

Proof. From Lemma A.1 there exists i_1 with $\lambda^*(i_1) > j \geqslant \kappa^*(\sigma(i_1))$. This implies $\sigma(i_1) \in I_\sigma(\lambda, \kappa)$ and thus $i_1 = i_o$. ∎

Lemma A.3. *Let $\lambda, \kappa \in \mathscr{T}_r(n)$ with $S_\lambda(k) = S_\kappa(k)$ and $k < m_\sigma$ or $k \geqslant M_\sigma$; then*

$$\lambda^*(i) \leqslant k \Leftrightarrow \kappa^*(\sigma(i)) \leqslant k.$$

Proof. $\# \{i \mid \lambda^*(i) \leqslant k\} = S_\lambda(k) = \# \{i \mid \kappa^*(\sigma(i)) \leqslant k\} = S_\kappa(k)$. We must show that the sets are equal. If not then there exist i_1, i_2 with $\lambda^*(i_1) > k \geqslant \kappa^*(\sigma(i_1))$ and $\lambda^*(i_2) \leqslant k < \kappa^*(\sigma(i_2))$. But then $\sigma(i_1) \in I_\sigma(\lambda, \kappa)$, so $i_1 = i_o$, which implies $m_\sigma \leqslant k < M_\sigma$, contradicting the conditions on k. ∎

Lemma A.4. *Let $\lambda, \kappa \in \mathscr{T}_r(n)$ denote, with $m \leqslant m_\sigma$, the greatest value for which $l_i = k_i$ for $i < m$ and, with $M \geqslant M_\sigma$, the smallest value for which $l_i = k_i$ for $i \geqslant M$; then*

$$\sigma(i) = i \qquad \textit{if} \quad \lambda^*(i) < m \quad \textit{or} \quad \lambda^*(i) \geqslant M > M_\sigma.$$

Moreover:

(i) *if* $l_m = k_m + 1$ *and* $I_\sigma(\lambda, \kappa) = \varnothing$ *then* $\lambda^*(i) = \kappa^*(\sigma(i))$ *for* $i < S_\lambda(m)$;

(ii) *if* $l_m = k_m$ *then* $\lambda^*(i) = \kappa^*(\sigma(i))$ *for* $i < S_\lambda(m_\sigma)$;

(iii) *if* $l_M = k_M$ *then* $\lambda^*(i) = \kappa^*(\sigma(i))$ *for* $i > i_o$.

Proof. From Lemma A.3 we obtain $k = \lambda^*(i) = \kappa^*(\sigma(i))$ if $k < m$ or $k \geqslant M > M_\sigma$. Since $\sigma \in S^{(\lambda)}$ this implies $\sigma(i) = i$ for these i.

(i) Remark that $\#\{i \mid \lambda^*(i) = m\} = l_m = \#\{i \mid \kappa^*(\sigma(i)) = m\} + 1$. So there is at least one i in the first set ($\lambda^*(i) = m$) that does not belong to the second set ($\kappa^*(\sigma(i)) \neq m$). In view of the first part one has $\kappa^*(\sigma(i)) > m$. Since $\sigma \in S^{(\lambda)}$ it is clear that the biggest $i = S_\lambda(m)$ satisfies these conditions. We must show now that this is the only i with this property. If not then there also exists i' with $\lambda^*(i') > m = \kappa^*(\sigma(i')) \Rightarrow I_\sigma(\lambda, \kappa) \neq \varnothing$.

(ii) Remark that $m = m_\sigma$. Since $\#\{i \mid \lambda^*(i) = m_\sigma\} = \#\{i \neq i_o \mid \kappa^*(\sigma(i)) = m_\sigma\} + 1$, we conclude as in (i) that $\lambda^*(i) = \kappa(\sigma(i))$ for $i < S_\lambda(m_\sigma)$.

(iii) Remark that $M = M_\sigma$. From Lemma A.3, $\#\{i \mid \lambda^*(i) \leqslant M_\sigma\} = \#\{i \mid \kappa^*(\sigma(i)) \leqslant M_\sigma\}$. Since i_0 is the only i with $\lambda^*(i) = M_\sigma > \kappa^*(\sigma(i))$, we conclude that $\lambda^*(i) = \kappa^*(\sigma(i)) = M_\sigma$ for $i > i_o$. ∎

In the following lemmas we simplify the calculation of $N_\sigma^+(\lambda, \kappa)$ and $N_\sigma^-(\lambda, \kappa)$. Therefore we introduce the following notation (replacing j by $\sigma(j)$):

$$N_\sigma^-(\lambda, \kappa) = \#\{(i, j) \mid i < j,\ \sigma(i) < \sigma(j) \text{ and } \lambda^*(i) > \kappa^*(\sigma(j))\}$$

$$N_\sigma^+(\lambda, \kappa)(i) = \#\{j < i \mid \sigma(i) < \sigma(j) \text{ and } \lambda^*(i) \leqslant \kappa^*(\sigma(j))\}.$$

Then

$$N_\sigma^+(\lambda, \kappa) = \sum_i N_\sigma^+(\lambda, \kappa)(i).$$

LEMMA A.5. (a) *If* (i, j) *contributes to* $N_\sigma^-(\lambda, \kappa)$ *then* $\sigma(i)$, $\sigma(j) \in I_\sigma(\lambda, \kappa)$.

(b) *If* $\# I_\sigma(\lambda, \kappa) \leqslant 1$ *then* $N_\sigma^-(\lambda, \kappa) = 0$.

(c) *If* $\sigma(i) \notin I_\sigma(\lambda, \kappa)$ *then* $N_\sigma^+(\lambda, \kappa)(i) = \#\{j < i \mid \sigma(i) < \sigma(j)\}$.

Proof. (a) follows directly from the definition of $N_\sigma^-(\lambda, \kappa)$ *and* (b) is a direct consequence of (a).

(c) For, if $\lambda^*(i) \leqslant \kappa^*(\sigma(i))$ then $\sigma(i) < \sigma(j)$ implies that $\lambda^*(i) \leqslant \kappa^*(\sigma(j))$, so this last condition can be omitted. ∎

LEMMA A.6. *Let $\lambda < \kappa$ and $S_\kappa(j) - 1 \leqslant S_\lambda(j) \leqslant S_\kappa(j)$ for every $j = 1, ..., n$. For $1 \leqslant m < r$ with $k_m > 0$ we consider $\kappa_m = (k_1, ..., k_m - 1, ..., k_r + 1) \in \mathscr{T}_r(n)$. For every b with $\kappa^*(b) = m$ there is a bijection,*

$$\phi\colon \{\sigma \in S^{(\lambda)} \mid I_\sigma(\lambda, \kappa) = \{b\}\} \to \{\sigma_o \in S^{(\lambda)} \mid I_{\sigma_o}(\lambda, \kappa_m) = \varnothing\}.$$

Moreover $C_b(\lambda, \kappa) = C_0(\lambda, \kappa_m)$.

Proof. Let s be the smallest value and $t - 1$ the greatest value of j for which $S_\kappa(j) - 1 = S_\lambda(j)$; i.e., $k_j = l_j$ for $j < s$ and $j > t$. Remark that $s < t$ exists since $\lambda < \kappa$. For $\sigma \in S^{(\lambda)}$ with $I_\sigma(\lambda, \kappa) = \{b\}$ Lemma A.2 yields $\lambda^*(i_o) = M_\sigma \geqslant t$. Define $\sigma_o = \phi(\sigma)$:

$$\begin{aligned} &\sigma_o(i) = \sigma(i) && \text{if} \quad \sigma(i) < b \quad \text{or} \quad \kappa^*(\sigma(i)) \geqslant M_\sigma \\ &\sigma_o(i) = \sigma(i) - 1 && \text{if} \quad b < \sigma(i) \quad \text{and} \quad \kappa^*(\sigma(i)) < M_\sigma. \\ &\sigma_o(i_o) = S_\kappa(M_\sigma - 1) && \end{aligned}$$

We prove now that ϕ is well defined and bijective. Remark first that

$$\lambda^*(i) \leqslant \kappa^*(\sigma(i)) \qquad \text{iff} \quad i \neq i_o. \tag{$*$}$$

Since $b = \sigma(i_o) < \sigma(i)$ for every i with $\lambda^*(i) = \lambda^*(i_0) = M_\sigma$ and $\sigma \in S^{(\lambda)}$ we conclude that i_o is the smallest value of i with $\lambda^*(i) = M_\sigma$:

$$i_o = S_\lambda(M_\sigma - 1) + 1. \tag{$**$}$$

Moreover

$$\text{If} \quad j \neq i \neq i_o \qquad \text{then} \quad \sigma(i) < \sigma(j) \Leftrightarrow \sigma_o(i) < \sigma_o(j). \tag{$***$}$$

Namely

$$\sigma_0(j) - \sigma_o(i) = \sigma(j) - \sigma(i) + \begin{cases} 1 \\ 0 \\ -1. \end{cases}$$

But $i \neq j$, so these expressions are both positive or both negative.

$*$ $\phi(\sigma) = \sigma_o \in S^{(\lambda)}$. I.e., $i < j$ and $\lambda^*(i) = \lambda^*(j)$; then $\sigma_0(i) < \sigma_o(j)$. If $i \neq j \neq i_o$ this follows from ($***$). If $\lambda^*(i) = \lambda^*(i_o) = M_\sigma$ then $i_o = S_\lambda(M_\sigma - 1) + 1 < i$ and ($*$) yields $\kappa^*(\sigma(i)) \geqslant M_\sigma$. We conclude that $\sigma_o(i) = \sigma(i) \geqslant S_\kappa(M_\sigma - 1) + 1 = \sigma_o(i_o)$.

$**$ $I_{\sigma_o}(\lambda, \kappa_m) = \varnothing$. Remark that κ^* and κ_m^* are related as follows:

$$\begin{aligned} \kappa^*(j) &= \kappa_m^*(j) && \text{if} \quad \kappa^*(j) < m \\ \kappa^*(j+1) &= \kappa_m^*(j) && \text{if} \quad \kappa^*(j) \geqslant m. \end{aligned}$$

We must show that $\lambda^*(i) \leqslant \kappa_m^*(\sigma_o(i))$ for every i.

(a) $\sigma_o(i) = \sigma(i) \notin I_\sigma(\lambda, \kappa)$. $\lambda^*(i) \leqslant \kappa^*(\sigma(i)) = \kappa^*(\sigma_o(i)) \leqslant \kappa_m^*(\sigma_o(i))$.

(b) $\sigma_o(i) + 1 = \sigma(i) \notin I_\sigma(\lambda, \kappa)$. Since $\kappa^*(\sigma(i)) \geqslant m$ we find

$$\lambda^*(i) \leqslant \kappa^*(\sigma(i)) = \kappa^*(\sigma_o(i) + 1) = \kappa_m^*(\sigma_o(i)).$$

(c) $i = i_o$. $\lambda^*(i_o) = M_\sigma = \kappa^*(S_\kappa(M_\sigma - 1) + 1) = \kappa_m^*(S_\kappa(M_\sigma - 1)) = \kappa_m^*(\sigma_o(i_o))$.

*** ϕ is bijective. I.e., every σ_o is the image of a unique σ. Since b is fixed, the only difficulty is to show that i_o is uniquely determined by σ_o. Using (**) it suffices to determine $M_\sigma = \lambda^*(i_o)$. Lemma A.4 (with $M = M_\sigma + 1$) implies that $\sigma(i) = i$ if $\lambda^*(i) > M_\sigma$. If $M_\sigma > t$ then $k_{M_\sigma} = l_{M_\sigma}$ and Lemma A.4(iii) yields $\lambda^*(i) = \kappa(\sigma(i))$ for $i > i_o$, and if $M_\sigma = t$ then $k_{M_\sigma} = l_{M_\sigma} + 1$. In both cases there are $L_{M_\sigma} - 1$ values of $i \neq i_o$ with $\lambda^*(i) = M_\sigma$, and $k_{M_\sigma} \geqslant l_{M_\sigma}$ values with $\kappa^*(\sigma(i)) = M_\sigma$. Therefore there also exists i' with $\lambda^*(i') < M_\sigma = \kappa^*(\sigma(i')) = \kappa^*(\sigma_o(i'))$. It follows from these considerations that M_σ is characterized as the greatest value of k for which there exists i with $\lambda^*(i) < \kappa^*(\sigma_o(i)) = k$.

**** $C_b(\lambda, \kappa) = C_0(\lambda, \kappa_m)$. Recall from Lemma A.5 that $N_\sigma^-(\lambda, \kappa) = N_{\sigma_o}^-(\lambda, \kappa_m) = 0$ and $N_\sigma^+(\lambda, \kappa)(i) = \#\{j < i \mid \sigma(i) < \sigma(j)\}$ if $\sigma(i) \notin I_\sigma(\lambda, \kappa)$. It is easy to see that i_o does not contribute to $N_\sigma^+(\lambda, \kappa)(i)$ or to $N_{\sigma_o}^+(\lambda, \kappa_m)(i)$, so using remark (**) we conclude for $i \neq i_o$ that $N_\sigma^+(\lambda, \kappa)(i) = N_{\sigma_o}^+(\lambda, \kappa_m)(i)$. For i_o we find

$$\begin{aligned} N_\sigma^+(\lambda, \kappa)(i_o) &= \#\{j < i_o \mid b < \sigma(j) \text{ and } M_\sigma \leqslant \kappa^*(\sigma(j))\} \\ &= \#\{j < i_o \mid b < \sigma_o(j) \text{ and } \sigma_o(j) > S_\kappa(M_\sigma - 1) = \sigma_o(i_o)\} \\ &= N_{\sigma_o}^+(\lambda, \kappa_m)(i_o). \end{aligned}$$

We conclude that $N_\sigma(\lambda, \kappa) = N_{\sigma_o}(\lambda, \kappa_m)$ and the statement clearly follows. ▌

We still need the following lemma concerning the permutations with at most one descent on a given set:

LEMMA A.7. *For* $1 \leqslant s \leqslant r$ *define* $B = \{i \mid \lambda^*(i) \leqslant s \leqslant \kappa^*(\sigma(i))\} = \{i_1 < \cdots < i_k\}$. *For* σ_0 *with no descent on* B, $\# I_{\sigma_0}(\lambda, \kappa) \leqslant 1$, $s \leqslant M_{\sigma_o}$, *and* $\sigma_o(j) \notin I_{\sigma_o}(\lambda, \kappa)$ *if* $j \in B$; *we define* $S_{\sigma_o} = \{\sigma \in S_n \mid \sigma(i) = \sigma_o(i)$ *if* $i \notin B$ *and* σ *has at most one descent* i_1 *on* $B\}$. *Then* $\sum_{\sigma \in S_{\sigma_o}} q^{N_\sigma(\lambda, \kappa)} = [k]\, q^{N_{\sigma_o}(\lambda, \kappa)}$.

Proof. In this proof we drop the specification (λ, κ) to simplify the notation. Remark that $\sigma \in S_{\sigma_o}$ is completely determined by $0 \leqslant l < k$, where $l+1$ is the greatest value with $\sigma(i_1) \geqslant \sigma(i_{l+1})$. We denote this σ by σ_l.

Remark that σ_o is indeed the given permutation. We find

$$\sigma_l(i_2) < \sigma_l(i_3) < \cdots < \sigma_l(i_{l+1}) < \sigma_l(i_1) < \sigma_l(i_{l+2}) < \cdots < \sigma_l(i_k).$$

Let us determine the relation between N_{σ_l} and N_{σ_o}. From Lemma A5(b) we find $N_\sigma^- = 0$ in both cases. To calculate N_σ^+ we remark: If $j \notin B$ then $\lambda^*(j) > s$ or $\kappa^*(\sigma_o(j)) < s$; this yields

$$j > i_k \qquad \text{or} \qquad \sigma_o(j) < \sigma_o(i_1). \tag{$*$}$$

From Lemma A.5(c) this yields that j does not contribute to $N_{\sigma_l}^+(i)$ for $i \in B$ and thus

$$\begin{aligned} N_{\sigma_l}^+(i_m) &= N_{\sigma_o}^+(i_m) = 0 && \text{if} \quad m = 1 \quad \text{or} \quad m > l+1 \\ N_{\sigma_l}^+(i_m) &= 1 = N_{\sigma_o}^+(i_m) + 1 && \text{if} \quad 1 < m \leqslant l+1. \end{aligned}$$

In order to relate $N_{\sigma_l}^+(i)$ with $N_{\sigma_o}^+(i)$ $(i \notin B)$ we investigate the condition

$$j < i, \qquad \sigma_l(j) > \sigma_l(i), \qquad \text{and} \qquad \lambda^*(i) \leqslant \kappa^*(\sigma_l(j). \tag{$**$}$$

If $i > i_k$ then for $j \in B$, $(**)$ is only a condition on $\sigma_l(j) = \sigma_o(j')$ for some $j' \in B$. We conclude that $N_{\sigma_l}^+(i) = N_{\sigma_o}^+(i)$.

From $(*)$ we can assume now that $\sigma_o(i) < \sigma_o(i_1)$ and $i \leqslant i_k$.

If $\sigma_o(i) \notin I_{\sigma_o}(\lambda, \kappa)$ then the last condition in $(**)$ can be ommited (cf. Lemma A.5(c)). Thus for $j \in B$, $(**)$ is only a condition on j. Again we conclude that $N_{\sigma_l}^+(i) = N_{\sigma_o}^+(i)$. This leaves the case $i = i_o \leqslant i_k$. Since $\lambda^*(i_o) = M_{\sigma_o} \geqslant s \geqslant \lambda^*(i_k)$ we conclude that $s = M_{\sigma_o}$. But then $\kappa^*(\sigma_l(j)) \geqslant M_{\sigma_o} \geqslant \lambda^*(i_0)$ for every $j \in B$. The last condition in $(**)$ is thus always satisfied for $j \in B$. We conclude that $(**)$ is only a condition on j and thus $N_{\sigma_l}^+(i_0) = N_{\sigma 0}^+(i_o)$. We resume $N_{\sigma_l}(\lambda, \kappa) = N_{\sigma_o}(\lambda, \kappa) + l$ and the formula follows. ∎

LEMMA A.8. *If* $I_\sigma(\lambda, \kappa) = \varnothing$ *and* $l_i = k_i$ *for* $i < m$, *and* $l_m = k_m + 1$ *or* $I_\sigma(\lambda, \kappa) = \{b\}$ *with* $\lambda = \kappa$, *define* $m = m_\sigma$; *then, for* $m < s \leqslant M_\sigma$,

$$B = \{i \mid \lambda^*(i) \leqslant s \leqslant \kappa^*(\sigma(i))\} = \{i \neq i_o \mid \lambda^*(i) = s\} \cup \{i_1\} \qquad \textit{with} \quad \lambda^*(i_1) < s.$$

Proof. Remark that $i_o \notin B$ and $\lambda^*(i_o) = M_\sigma \geqslant s$; therefore

$$\#\{i \neq i_o \mid \kappa^*(\sigma(i)) \geqslant s\} = \#\{i \neq i_o \mid \lambda^*(i) \geqslant s\} + 1.$$

So there exists at least one $i_1 \in B$ with $\kappa^*(\sigma(i_1)) \geqslant s > \lambda^*(i_1)$. We must show that this is the only $i \in B$ with this property. If not then there also exists $i' \neq i_o$ with $\lambda^*(i') \geqslant s > \kappa^*$ $(\sigma(i'))$, contradicting the conditions on $I_\sigma(\lambda, \kappa)$. ∎

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